

Eigenvalues of the $p(x)$ -Laplacian Steklov problem [☆]

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Abstract

Consider Steklov eigenvalue problem involving the $p(x)$ -Laplacian on a bounded domain Ω , the open subset of \mathbb{R}^N with $N \geq 2$, as follows

$$\begin{cases} \Delta_{p(x)} u = |u|^{p(x)-2} u & \text{in } \Omega, \\ |\nabla u|^{p(x)-2} \frac{\partial u}{\partial \gamma} = \lambda |u|^{p(x)-2} u & \text{on } \partial\Omega, \end{cases}$$

where $p(x) \neq \text{constant}$.

We prove that the existence of infinitely many eigenvalue sequences. Unlike the p -Laplacian case, for a variable exponent $p(x)$ ($\neq \text{constant}$), there does not exist a principal eigenvalue and the set of all eigenvalues is not closed under some assumptions. Finally, we present some sufficient conditions for the infimum of all eigenvalues is zero and positive, respectively.

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1. Introduction

Nonlinear eigenvalue problems for the p -Laplacian subject to different kinds of boundary conditions on a bounded domain have been studied extensively and many interesting results have been obtained. For example, see [2–4,6,9,10,13,14,24,25,28,29,32,34] and references therein. The investigations mainly have relied on variational methods and deduce the existence of a principal eigenvalue, which is the smallest of all possible eigenvalues, as a consequence of minimization results of appropriate functionals.

For the p -Laplacian Dirichlet eigenvalue problem, many results have been obtained. For example, the spectrum of the Dirichlet problem has, among others, the following properties:

- (1) There exists a nondecreasing sequence of nonnegative eigenvalues $\{\lambda_n\}$ tending to ∞ as $n \rightarrow \infty$ (see [4]).
- (2) The first eigenvalue λ_1 is simple and only eigenfunctions associated with λ_1 do not change sign (see [2,28]).

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- (3) The set of eigenvalues is closed (see [3]).
- (4) The first eigenvalue λ_1 is isolated (see [28]).
- (5) The eigenvalue λ_2 is the second eigenvalue (see [3]), i.e.

$$\lambda_2 = \inf\{\lambda: \lambda \text{ is an eigenvalue of the Dirichlet problem and } \lambda > \lambda_1\}.$$

For the p -Laplacian no-flux, Neumann, Robin and Steklov eigenvalue problems, by using a unified treatment, the above five properties, among other things, also hold (see [29]).

Comparatively, nonlinear eigenvalue problems for the $p(x)$ -Laplacian have been investigated little. The author, Prof. Xianling Fan, has studied the eigenvalue problem for the $p(x)$ -Laplacian subject to zero Neumann boundary conditions on a bounded domain (see [17]), and the eigenvalues of the $p(x)$ -Laplacian Dirichlet problem have been investigated by Fan, Zhang and Zhao (see [22]). The authors, M. Mihailescu and V. Radulescu, have studied nonhomogeneous quasilinear eigenvalue problem with variable exponent (see [30]).

The operator $-\Delta_{p(x)}u = -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ with $p(x) > 1$ is called the $p(x)$ -Laplacian which is a natural generalization of the p -Laplacian (where $p > 1$ is a constant). When $p(x) \not\equiv \text{constant}$, the $p(x)$ -Laplacian possesses more complicated nonlinearity than the p -Laplacian, say, it is inhomogeneous. For this reason, some of the above properties of the p -Laplacian eigenvalue problems may not hold for a general $p(x)$ -Laplacian case. By now the following properties, which are different from the p -Laplacian case, have been obtained:

- (1) For the $p(x)$ -Laplacian eigenvalue problems with $p(x) \not\equiv \text{constant}$, there exist infinitely many eigenvalue sequences $\{\lambda_{(n,\alpha)}\}$ tending to ∞ as $n \rightarrow \infty$ (see Section 3 and [17,22]).
- (2) In [22] and Section 4, for the $p(x)$ -Laplacian Dirichlet and Steklov eigenvalue problems, under some assumptions the infimum of all eigenvalues is zero. This means that under some conditions there does not exist a principal eigenvalue and the set of eigenvalues is not closed.
- (3) For the $p(x)$ -Laplacian Neumann eigenvalue problem, the smallest eigenvalue of the problem, λ_1 , is zero and simple, all eigenfunctions associated with λ_1 are nonzero constant functions, but under some assumptions the first eigenvalue is not isolated, that is, the infimum of all positive eigenvalue of the problem is zero. It means that there does not exist the second eigenvalue under some conditions (see [17]).

Besides being of mathematical interest, the study of the $p(x)$ -Laplace operator is also of interest both in nonlinear elasticity theory and in electrorheological fluids (see [11,12,33,38–41]).

In this paper, we will deal with the following Steklov eigenvalue problem involving the $p(x)$ -Laplacian which is a new topic

$$(S) \quad \begin{cases} \Delta_{p(x)}u = |u|^{p(x)-2}u & \text{in } \Omega, \\ |\nabla u|^{p(x)-2} \frac{\partial u}{\partial \gamma} = \lambda |u|^{p(x)-2}u & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in \mathbb{R}^N with $N \geq 2$, the boundary $\partial\Omega$ is C^1 , γ is the outward unit normal to $\partial\Omega$, $p \in C(\overline{\Omega}, \mathbb{R})$ with $p(x) > 1$ and $p(x) \not\equiv \text{constant}$, $\lambda \in \mathbb{R}$. we prove that the existence of infinitely many eigenvalue sequences for the problem (S) (see Section 3) and also present some sufficient conditions for the infimum of all eigenvalues of the problem (S) is zero and positive, respectively (see Section 4). In order to obtain these results, in Section 2, we state some elementary properties of the space $W^{1,p(x)}(\Omega)$. In particular, we prove a weighted variable exponent Sobolev trace compact embedding theorem which will be useful later.

2. Space $W^{1,p(x)}(\Omega)$ and weighted trace theorem

In order to deal with the problem (S), we need some theory of variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ (see [20,23,27]). For convenience, we only recall some basic facts which will be used later.

Suppose that Ω is a bounded domain of \mathbb{R}^N with a smooth boundary $\partial\Omega$, and $p \in C(\overline{\Omega}, \mathbb{R})$ with $p(x) > 1$. Denote by $p^- := \inf_{x \in \Omega} p(x)$ and $p^+ := \sup_{x \in \Omega} p(x)$, then, $p^- > 1$ and $p^+ < +\infty$.

Define the variable exponent Lebesgue space $L^{p(x)}(\Omega)$ by

$$L^{p(x)}(\Omega) = \left\{ u \mid u : \Omega \rightarrow \mathbb{R} \text{ is a measurable and } \int_{\Omega} |u(x)|^{p(x)} dx < +\infty \right\}$$

with the norm

$$|u|_{p(x)} = |u|_{L^{p(x)}(\Omega)} = \inf \left\{ \tau > 0 : \int_{\Omega} \left| \frac{u(x)}{\tau} \right|^{p(x)} dx \leq 1 \right\}.$$

Define the variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ by

$$W^{1,p(x)}(\Omega) = \{ u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega) \}$$

with the norm

$$\|u\| = \inf \left\{ \tau > 0 : \int_{\Omega} \left[\left| \frac{\nabla u}{\tau} \right|^{p(x)} + \left| \frac{u}{\tau} \right|^{p(x)} \right] dx \leq 1 \right\}.$$

Proposition 2.1. (See [16,18,19,23].) Both $(L^{p(x)}(\Omega), |\cdot|_{p(x)})$ and $(W^{1,p(x)}(\Omega), \|\cdot\|)$ are separable, reflexive and uniformly convex Banach spaces.

Proposition 2.2. (See [8,16,19,21,23].) Hölder inequality holds, namely,

$$\int_{\Omega} |uv| dx \leq 2|u|_{p(x)} |v|_{q(x)}, \quad \forall u \in L^{p(x)}(\Omega), \quad \forall v \in L^{q(x)}(\Omega),$$

where $\frac{1}{p(x)} + \frac{1}{q(x)} = 1$.

Proposition 2.3. (See [19].) Let $\varrho(u) = \int_{\Omega} [|\nabla u|^{p(x)} + |u|^{p(x)}] dx$. For $u, u_k \in W^{1,p(x)}(\Omega)$ ($k = 1, 2, \dots$), we have

- (1) $\|u\| \leq 1 \Rightarrow \|u\|^{p^+} \leq \varrho(u) \leq \|u\|^{p^-}$;
- (2) $\|u\| \geq 1 \Rightarrow \|u\|^{p^-} \leq \varrho(u) \leq \|u\|^{p^+}$;
- (3) $\|u_k\| \rightarrow 0 \Leftrightarrow \varrho(u_k) \rightarrow 0$;
- (4) $\|u_k\| \rightarrow \infty \Leftrightarrow \varrho(u_k) \rightarrow \infty$.

Let $a : \partial\Omega \rightarrow \mathbb{R}$ be a measurable.

Define the weighted variable exponent Lebesgue space by

$$L_{a(x)}^{p(x)}(\partial\Omega) = \left\{ u \mid u : \partial\Omega \rightarrow \mathbb{R} \text{ is a measurable and } \int_{\partial\Omega} |a(x)| |u(x)|^{p(x)} d\sigma_x < +\infty \right\}$$

with the norm

$$|u|_{(p(x), a(x))} = |u|_{L_{a(x)}^{p(x)}(\partial\Omega)} = \inf \left\{ \tau > 0 : \int_{\partial\Omega} |a(x)| \left| \frac{u(x)}{\tau} \right|^{p(x)} d\sigma_x \leq 1 \right\},$$

where $d\sigma_x$ is the measure on the boundary. Then, $L_{a(x)}^{p(x)}(\partial\Omega)$ is a Banach space. In particular, when $a(x) \equiv 1$ on $\partial\Omega$, $L_{a(x)}^{p(x)}(\partial\Omega) = L^{p(x)}(\partial\Omega)$.

Proposition 2.4. Let $\rho(u) = \int_{\partial\Omega} |a(x)| |u(x)|^{p(x)} d\sigma_x$. For $u, u_k \in L_{a(x)}^{p(x)}(\partial\Omega)$ ($k = 1, 2, \dots$), we have

- (1) $|u|_{(p(x), a(x))} \geq 1 \Rightarrow |u|_{(p(x), a(x))}^{p^-} \leq \rho(u) \leq |u|_{(p(x), a(x))}^{p^+}$;
- (2) $|u|_{(p(x), a(x))} \leq 1 \Rightarrow |u|_{(p(x), a(x))}^{p^+} \leq \rho(u) \leq |u|_{(p(x), a(x))}^{p^-}$;
- (3) $|u_k|_{(p(x), a(x))} \rightarrow 0 \Leftrightarrow \rho(u_k) \rightarrow 0$;
- (4) $|u_k|_{(p(x), a(x))} \rightarrow \infty \Leftrightarrow \rho(u_k) \rightarrow \infty$.

For $A \subset \overline{\Omega}$, denote by $p^-(A) = \inf_{x \in A} p(x)$, $p^+(A) = \sup_{x \in A} p(x)$. Define

$$p^\partial(x) = (p(x))^\partial := \begin{cases} \frac{(N-1)p(x)}{N-p(x)}, & \text{if } p(x) < N, \\ \infty, & \text{if } p(x) \geq N, \end{cases}$$

$$p_{r(x)}^\partial(x) := \frac{r(x) - 1}{r(x)} p^\partial(x),$$

where $x \in \partial\Omega$, $r \in C(\partial\Omega, \mathbb{R})$ with $r^- =: \inf_{x \in \partial\Omega} r(x) > 1$.

Theorem 2.1. Assume that the boundary of Ω possesses the cone property and $p \in C(\overline{\Omega})$ with $p^- > 1$. Suppose that $a \in L^{r(x)}(\partial\Omega)$, $r \in C(\partial\Omega)$ with $r(x) > \frac{p^\partial(x)}{p^\partial(x)-1}$ for all $x \in \partial\Omega$. If $q \in C(\partial\Omega)$ and

$$1 \leq q(x) < p_{r(x)}^\partial(x), \quad \forall x \in \partial\Omega. \quad (2.1)$$

Then, there exists a compact embedding $W^{1,p(x)}(\Omega) \hookrightarrow L_{a(x)}^{q(x)}(\partial\Omega)$. In particular, there is a compact embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q_0(x)}(\partial\Omega)$ where $1 \leq q_0(x) < p^\partial(x)$, $\forall x \in \partial\Omega$.

Proof. Put $h(x) = \frac{r(x)}{r(x)-1} q(x) := r_0(x)q(x)$. Let $v \in L^{h(x)}(\partial\Omega)$, it is clear that $|v|^{q(x)} \in L^{r_0(x)}(\partial\Omega)$, employing Hölder inequality, we get

$$\int_{\partial\Omega} |a(x)| |v|^{q(x)} d\sigma_x \leq 2|a|_{L^{r(x)}(\partial\Omega)} \| |v|^{q(x)} \|_{L^{r_0(x)}(\partial\Omega)} < +\infty.$$

Hence $v \in L_{a(x)}^{q(x)}(\partial\Omega)$, i.e. $L^{h(x)}(\partial\Omega) \subset L_{a(x)}^{q(x)}(\partial\Omega)$.

It follows from (2.1) that $h(x) < p^\partial(x)$. Then for each given $\bar{x} \in \partial\Omega$, there is a relatively open neighborhood $\Omega(\bar{x})$ of \bar{x} in $\overline{\Omega}$ such that

$$h^+(\partial\Omega \cap \Omega(\bar{x})) < (p^-(\Omega(\bar{x})))^\partial.$$

By the constant exponent Sobolev trace compact embedding theorem (see [1,15]), we have the compact embedding

$$W^{1,p^-(\Omega(\bar{x}))}(\Omega(\bar{x})) \hookrightarrow L^{h^+(\partial\Omega \cap \Omega(\bar{x}))}(\partial\Omega \cap \Omega(\bar{x})).$$

Since $W^{1,p(x)}(\Omega(\bar{x})) \subset W^{1,p^-(\Omega(\bar{x}))}(\Omega(\bar{x}))$ and $L^{h^+(\partial\Omega \cap \Omega(\bar{x}))}(\partial\Omega \cap \Omega(\bar{x})) \subset L^{h(x)}(\partial\Omega \cap \Omega(\bar{x}))$, the embedding $W^{1,p(x)}(\Omega(\bar{x})) \hookrightarrow L_{a(x)}^{q(x)}(\partial\Omega \cap \Omega(\bar{x}))$ is compact.

Applying the finite covering theorem to the compact set $\partial\Omega$, we can obtain the embedding $W^{1,p(x)}(\Omega) \hookrightarrow L_{a(x)}^{q(x)}(\partial\Omega)$ is compact. \square

3. Existence of infinitely many eigenvalue sequences

In this section and next section, for brevity we write $X = W^{1,p(x)}(\Omega)$. Denote by $u_n \rightharpoonup u$ and $u_n \rightarrow u$ the weak convergence and strong convergence of sequence $\{u_n\}$ in X , respectively.

Definition 3.1.

(1) A pair $(u, \lambda) \in X \times \mathbb{R}$ is a weak solution of the Steklov problem (S) provided that for any $v \in X$,

$$\int_{\Omega} |\nabla u|^{p(x)-2} \nabla u \nabla v \, dx + \int_{\Omega} |u|^{p(x)-2} uv \, dx = \lambda \int_{\partial\Omega} |u|^{p(x)-2} uv \, d\sigma_x. \quad (3.1)$$

(2) Such a pair $(u, \lambda) \in X \times \mathbb{R}$, with u nontrivial, is called an eigenpair, λ is an eigenvalue and u is called an associated eigenfunction.

For any $u \in X$, define $F, G : X \rightarrow \mathbb{R}$ by

$$F(u) = \int_{\Omega} \frac{1}{p(x)} [|\nabla u|^{p(x)} + |u|^{p(x)}] \, dx, \quad G(u) = \int_{\partial\Omega} \frac{1}{p(x)} |u|^{p(x)} \, d\sigma_x,$$

where $d\sigma_x$ is the measure on the boundary.

Then it is easy to see that the functional $F : X \rightarrow \mathbb{R}$ is coercive, convex and sequentially weakly lower semi-continuous. By Theorem 2.1, $G : X \rightarrow \mathbb{R}$ is sequentially weakly-strongly continuous, namely, $u_n \rightharpoonup u_0$ in X implies $G(u_n) \rightarrow G(u_0)$. $F, G \in C^1(X, \mathbb{R})$, and denote by $A(u), B(u)$ the derivatives of F, G at $u \in X$, i.e. for any $u, v \in X$,

$$\begin{aligned} \langle A(u), v \rangle &= \langle F'(u), v \rangle = \int_{\Omega} |\nabla u|^{p(x)-2} \nabla u \nabla v \, dx + \int_{\Omega} |u|^{p(x)-2} uv \, dx, \\ \langle B(u), v \rangle &= \langle G'(u), v \rangle = \int_{\partial\Omega} |u|^{p(x)-2} uv \, d\sigma_x. \end{aligned}$$

Below we give several technical results that will be used later.

Proposition 3.1. (See [17,18,21].)

- (1) The mapping $A : X \rightarrow X^*$ is a strictly monotone, bounded homeomorphism (i.e. A and A^{-1} are continuous), and is of type (S_+) , namely: $u_n \rightharpoonup u_0$ in X and $\limsup_{n \rightarrow \infty} A(u_n)(u_n - u_0) \leq 0$ imply $u_n \rightarrow u_0$ in X .
 (2) A^{-1} is bounded.

Applying Theorem 2.1, we have

Proposition 3.2. $B : X \rightarrow X^*$ is sequentially weakly-strongly continuous, namely, $u_n \rightharpoonup u_0$ in X implies $B(u_n) \rightarrow B(u_0)$ in X^* .

It is clear that $(u, \lambda) \in X \times \mathbb{R}$ is a weak solution of the Steklov eigenvalue problem (S) if and only if

$$A(u) = \lambda B(u). \quad (3.2)$$

Remark 3.1. If u is an eigenfunction associated with λ , then

$$\frac{p^+ F(u)}{p^- G(u)} \geq \lambda = \frac{\langle F'(u), u \rangle}{\langle G'(u), u \rangle} \geq \frac{p^- F(u)}{p^+ G(u)} > 0.$$

In order to solve the eigenvalue problem (3.2), in general, the constrained variational method is employed (see [4, 7,13,14,29,35,36]). In this paper, we take F as an objective functional and G as a constraint functional.

For any $\alpha > 0$, denote by

$$M_\alpha = \{u \in X : G(u) = \alpha\}.$$

For any $u \in M_\alpha$, $\langle G'(u), u \rangle = \int_{\partial\Omega} |u|^{p(x)} \, d\sigma_x \geq p^- \alpha > 0$, hence M_α is a C^1 -submanifold of X with codimension one. And by Theorem 2.1, M_α is sequentially weakly closed.

It is well known that $(u, \lambda) \in X \times \mathbb{R}$ solves (3.2) if and only if u is a critical point of F with respect to M_α (see [36]).

Denote by $T_u(M_\alpha)$ the tangent space at $u \in M_\alpha$, i.e.

$$T_u(M_\alpha) = \ker(G'(u)) = \{v \in X : \langle G'(u), v \rangle = 0\}.$$

Let $\tilde{F} = F|_{M_\alpha} : M_\alpha \rightarrow \mathbb{R}$ the restriction of F on M_α and $d\tilde{F} = F'|_{T_u(M_\alpha)}$ the derivative of \tilde{F} on M_α , i.e. the restriction of A on $T_u(M_\alpha)$.

Lemma 3.1. *For any $\alpha > 0$, the functional $\tilde{F} : M_\alpha \rightarrow \mathbb{R}$ satisfies (PS) condition, namely, any sequence $\{u_n\} \subset M_\alpha$ such that $\tilde{F}(u_n) \rightarrow c$ and $d\tilde{F}(u_n) \rightarrow 0$ contains a converging subsequence.*

Proof. Let $u \in M_\alpha$ and $w = A^{-1}(B(u))$, then $B(u) \neq 0$ and $w \neq 0$. We have

$$\langle B(u), A^{-1}(B(u)) \rangle = \langle A(w), w \rangle = \int_{\Omega} |\nabla w|^{p(x)} dx + \int_{\Omega} |w|^{p(x)} dx > 0.$$

Hence, $w = A^{-1}(B(u)) \notin T_u(M_\alpha)$. We have

$$X = T_u(M_\alpha) \oplus \{\beta A^{-1}(B(u)) : \beta \in \mathbb{R}\}. \quad (3.3)$$

Let $P : X \rightarrow T_u(M_\alpha)$ be the natural projection. Then, for every $v \in X$, there exists a unique $\beta \in \mathbb{R}$ such that

$$v = Pv + \beta A^{-1}(B(u)).$$

We have $\langle B(u), Pv \rangle = 0$ and

$$\langle B(u), v \rangle = \langle B(u), Pv + \beta A^{-1}(B(u)) \rangle = \beta \langle B(u), A^{-1}(B(u)) \rangle.$$

Thus,

$$\beta = \frac{\langle B(u), v \rangle}{\langle B(u), A^{-1}(B(u)) \rangle}.$$

Consequently,

$$\begin{aligned} \langle d\tilde{F}(u), v \rangle &= \langle A(u), Pv \rangle \\ &= \langle A(u), v \rangle - \left\langle A(u), \frac{\langle B(u), v \rangle}{\langle B(u), A^{-1}(B(u)) \rangle} A^{-1}(B(u)) \right\rangle \\ &= \langle A(u), v \rangle - \frac{\langle A(u), A^{-1}(B(u)) \rangle}{\langle B(u), A^{-1}(B(u)) \rangle} \langle B(u), v \rangle \\ &= \left\langle A(u) - \frac{\langle A(u), A^{-1}(B(u)) \rangle}{\langle B(u), A^{-1}(B(u)) \rangle} B(u), v \right\rangle, \end{aligned}$$

which shows that

$$d\tilde{F}(u) = A(u) - \frac{\langle A(u), A^{-1}(B(u)) \rangle}{\langle B(u), A^{-1}(B(u)) \rangle} B(u) = A(u) - \lambda(u) B(u),$$

where

$$\lambda(u) = \frac{\langle A(u), A^{-1}(B(u)) \rangle}{\langle B(u), A^{-1}(B(u)) \rangle}. \quad (3.4)$$

Since $\{u_n\} \subset M_\alpha$ is such that $\tilde{F}(u_n) \rightarrow c$ and F is coercive, $\{\|u_n\|\}$ is bounded.

By the reflexivity of X , there exist $u_0 \in X$ and a subsequence of $\{u_n\}$ weakly converging to u_0 in X . We still denote by $\{u_n\}$, i.e. $u_n \rightharpoonup u_0$ in X . Applying Proposition 3.2 and Theorem 2.1, we have $B(u_n) \rightarrow B(u_0)$ in X^* and $G(u_n) \rightarrow G(u_0)$. Hence, $u_0 \in M_\alpha$.

Let $w_n = A^{-1}(B(u_n))$, since $B(u_n) \rightarrow B(u_0) \neq 0$ in X^* , then $w_n \rightarrow w_0 \neq 0$ in X and

$$\langle B(u_n), A^{-1}(B(u_n)) \rangle = \langle A(w_n), w_n \rangle \rightarrow \int_{\Omega} |\nabla w_0|^{p(x)} dx + \int_{\Omega} |w_0|^{p(x)} dx > 0,$$

$$|\langle A(u_n), A^{-1}(B(u_n)) \rangle| = |\langle A(u_n), w_n \rangle| \leq c_1 \|u_n\| \|w_n\| < c_2.$$

And by (3.4), we can conclude that $\{\lambda(u_n)\}$ is bounded. We may assume, taking a subsequence if necessary, that $\lambda(u_n) \rightarrow \lambda_0$. Since $d\tilde{F}(u_n) \rightarrow 0$,

$$u_n \rightarrow A^{-1}(\lambda_0 B(u_0)). \quad \square$$

Put

$$\Sigma_{\alpha} = \{H \subset M_{\alpha} : H \text{ is compact and } -H = H\}.$$

Denote by $\gamma(H)$ the genus of $H \in \Sigma_{\alpha}$ (see [7,36]). Define

$$c_{(n,\alpha)} = \inf_{H \in \Sigma_{\alpha}} \sup_{u \in H, \gamma(H) \geq n} \tilde{F}(u) \quad (n = 1, 2, \dots). \quad (3.5)$$

Proposition 3.3 (*Ljusternik–Schnirelmann principle*). (See [35, Corollary 4.1].) Suppose that M is a closed symmetric C^1 -submanifold of a real Banach space X and $0 \notin M$. Suppose also that $f \in C^1(M, \mathbb{R})$ is even and bounded below. Define

$$c_j = \inf_{H \in \Gamma_j} \sup_{x \in H} f(x),$$

where $\Gamma_j = \{H \subset M : \gamma(H) \geq j, -H = H \text{ and } H \text{ is compact}\}$. If $\Gamma_k \neq \emptyset$ for some $k \geq 1$ and if f satisfies $(PS)_c$ for all $c = c_j, j = 1, 2, \dots, k$, then f has at least k distinct pairs of critical points.

Remark 3.2. In [35], it is proved that all c_j are critical values of f and

$$-\infty < c_1 \leq c_2 \leq \dots \leq c_k < +\infty.$$

Since X is a separable and reflexive Banach space (see [16,37]), there exist $\{e_n\}_{n=1}^{\infty} \subset X$ and $\{f_n\}_{n=1}^{\infty} \subset X^*$ such that

$$f_n(e_m) = \delta(n, m) = \begin{cases} 1 & \text{if } n = m, \\ 0 & \text{if } n \neq m, \end{cases}$$

$$X = \overline{\text{span}}\{e_n : n = 1, 2, \dots\}, \quad X^* = \overline{\text{span}}^{W^*}\{f_n : n = 1, 2, \dots\}.$$

For $k = 1, 2, \dots$, denote by

$$X_n = \text{span}\{e_n\}, \quad Y_n = \bigoplus_{j=1}^n X_j, \quad Z_n = \overline{\bigoplus_{j=n}^{\infty} X_j}.$$

Proposition 3.4. (See [5].) Assume that $\Psi : X \rightarrow \mathbb{R}$ is weakly-strongly continuous and $\Psi(0) = 0$. Let $r > 0$ be given. Then,

$$\lim_{n \rightarrow \infty} \sup_{u \in Z_n, \|u\| \leq r} |\Psi(u)| = 0.$$

Lemma 3.2. $\lim_{n \rightarrow \infty} \inf_{u \in Z_n \cap M_{\alpha}} \|u\| = +\infty$.

Proof. Arguing by contradiction, suppose that there exist $c_0 > 0$ and $\{u_n\} \subset Z_n \cap M_\alpha$ such that $\|u_n\| \leq c_0, \forall n \in \mathbb{N}$. Then,

$$\lim_{n \rightarrow \infty} \sup_{u \in Z_n, \|u\| \leq c_0} |G(u)| \geq \lim_{n \rightarrow \infty} |G(u_n)| = \alpha > 0.$$

By Proposition 3.4, it is a contradiction. \square

Lemma 3.3. $\lim_{n \rightarrow \infty} c(n, \alpha) = +\infty$.

Proof. By Lemma 3.2, for each $c > 1$, there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$ and $u \in Z_n \cap M_\alpha, \|u\| > c$. And for any $H \in \Sigma_\alpha, \gamma(H \cap Y_{n-1}) \leq n - 1$. On the other hand, the codimension of $Z_n \leq n - 1$, by the property of genus (see [31,35]), for $H \in \Sigma_\alpha$ with $\gamma(H) \geq n, H \cap Z_n$ is nonempty. Then,

$$\begin{aligned} c(n, \alpha) &= \inf_{H \in \Sigma_\alpha} \sup_{u \in H, \gamma(H) \geq n} \tilde{F}(u) \\ &= \inf_{H \in \Sigma_\alpha} \max \left\{ \sup_{u \in H \cap (X \setminus Y_{n-1}), \gamma(H) \geq n} \tilde{F}(u), \sup_{u \in H \cap Y_{n-1}, \gamma(H) \geq n} \tilde{F}(u) \right\} \\ &= \inf_{H \in \Sigma_\alpha} \max \left\{ \sup_{u \in H \cap ((X \setminus Y_{n-1}) \setminus Z_n), \gamma(H) \geq n} \tilde{F}(u), \sup_{u \in H \cap Z_n, \gamma(H) \geq n} \tilde{F}(u) \right\} \\ &\geq \inf_{H \in \Sigma_\alpha} \sup_{u \in H \cap Z_n, \gamma(H) \geq n} \tilde{F}(u) \geq \inf_{H \in \Sigma_\alpha} \sup_{u \in H \cap Z_n, \gamma(H) \geq n} \frac{\|u\|^{p^-}}{p^+} \geq \frac{c^{p^-}}{p^+}. \quad \square \end{aligned}$$

Applying Lemma 3.1, Ljusternik–Schnirelmann Principle and Lemma 3.3 to the Steklov problem (S), we can easily obtain the following theorem.

Theorem 3.1. For each $n \in \mathbb{N}$ and $\alpha > 0$, $c(n, \alpha)$ defined by (3.5) is a critical value of \tilde{F} on submanifold M_α such that

$$0 < c(n, \alpha) \leq c(n+1, \alpha), \quad c(n, \alpha) \rightarrow +\infty \quad \text{as } n \rightarrow \infty.$$

Moreover, the Steklov problem (S) has infinitely many eigenpair sequences $\{(u_{(n, \alpha)}, \lambda_{(n, \alpha)})\}$ such that

$$\begin{aligned} G(\pm u_{(n, \alpha)}) &= \alpha, \quad F(\pm u_{(n, \alpha)}) = c(n, \alpha), \\ \lambda_{(n, \alpha)} &= \frac{\langle F'(u_{(n, \alpha)}), u_{(n, \alpha)} \rangle}{\langle G'(u_{(n, \alpha)}), u_{(n, \alpha)} \rangle} \rightarrow +\infty \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Remark 3.3.

- (1) It is well known that (see [24,29,32]), in the case of $p(x) \equiv p = \text{constant}$, owing to the homogeneity, the values $\frac{c(n, \alpha)}{\alpha}$ are independent of $\alpha > 0$, so are $\lambda_{(n, \alpha)}$, in particular, $\lambda_1 = \lambda_{(1, \alpha)} > 0$. However, for a general variable exponent $p(x)$ ($\neq \text{constant}$), this is not true owing to the loss of the homogeneity. Hence, for the Steklov problem (S), there are infinitely many eigenvalue sequences $\{\lambda_{(n, \alpha)}\}$.
- (2) We do not know whether the Steklov problem (S) only has eigenvalue sequences $\{\lambda_{(n, \alpha)}\}$.

4. The infimum of all the eigenvalues

In this section, we will discuss the infimum of all eigenvalues of the problem (S) and present some sufficient conditions for the infimum of all the eigenvalues is zero and positive, respectively.

Define

$$\Lambda = \{\lambda: \lambda \text{ is an eigenvalue of the Steklov problem (S)}\}, \quad \lambda_* = \inf \Lambda.$$

For $A \subset \overline{\Omega}$ and $\delta > 0$, denote by $B(A, \delta) = \{x \in \mathbb{R}^N: \text{dist}(x, A) < \delta\}$, $B_\Omega(A, \delta) = B(A, \delta) \cap \Omega$, $B_{\partial\Omega}(A, \delta) = B(A, \delta) \cap \partial\Omega$.

Lemma 4.1. For any $\delta, \alpha > 0$, let $\beta(u) = \int_{B_\Omega(\partial\Omega, \delta)} [|\nabla u|^{p(x)} + |u|^{p(x)}] dx$. Then, $\beta_{(\delta, \alpha)} =: \inf_{u \in M_\alpha} \beta(u) > 0$.

Proof. Arguing by contradiction, assume that $\beta_{(\delta, \alpha)} = 0$, then there exists a sequence $\{u_n\} \subset M_\alpha$ such that $\beta(u_n) \rightarrow 0$ as $n \rightarrow \infty$. By Proposition 2.3, we have

$$\|u_n\|_{W^{1,p(x)}(B_\Omega(\partial\Omega, \delta))} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

On the other hand, since $\int_{\partial B_\Omega(\partial\Omega, \delta)} |u_n|^{p(x)} d\sigma_x \geq \int_{\partial\Omega} |u_n|^{p(x)} d\sigma_x = \alpha$, and by Proposition 2.4, we have

$$|u_n|_{L^{p(x)}(\partial B_\Omega(\partial\Omega, \delta))} \geq \min\left\{\alpha^{\frac{1}{p^+}}, \alpha^{\frac{1}{p^-}}\right\} > 0.$$

According to Theorem 2.1, this is a contradiction. \square

Lemma 4.2. For all $\alpha > 0$, let u_0 be an eigenfunction associated with $\lambda_{(1, \alpha)}$. Then,

$$F(u_0) = c_{(1, \alpha)} = \inf\{F(u) : u \in M_\alpha\}.$$

Proof. Let $b_\alpha = \inf\{F(u) : u \in M_\alpha\}$. Obviously, $b_\alpha \leq c_{(1, \alpha)}$.

Since the functional $F : X \rightarrow \mathbb{R}$ is coercive and sequentially weakly lower semi-continuous and M_α is a sequentially weakly closed subset of X , there exists $u_* \in M_\alpha$ such that $F(\pm u_*) = b_\alpha$. Let $H = \{\pm u_*\}$, then $\gamma(H) = 1$ and $c_{(1, \alpha)} \leq b_\alpha$. \square

Remark 4.1. Let u_n be the eigenfunction associated with $\lambda_{(n, \alpha)}$, by Remark 3.1, we have

$$\lambda_{(n, \alpha)} \geq \frac{p^- F(u_n)}{p^+ G(u_n)} \geq \frac{p^- c_{(1, \alpha)}}{p^+ \alpha} \geq \left(\frac{p^-}{p^+}\right)^2 \lambda_{(1, \alpha)}.$$

Theorem 4.1. Suppose that there exists $\delta > 0$ such that for all $x \in B_\Omega(\partial\Omega, \delta)$, $p(x) \equiv p = \text{constant}$. Then, $\lambda_* > 0$.

Proof. Let u be the eigenfunction associated with λ . Then, $G(u) \neq 0$ and there exists $t > 0$ such that $u_1 = \frac{u}{t} \in M_1$. We have

$$\lambda \geq \frac{\int_{B_\Omega(\partial\Omega, \delta)} [|\nabla u|^{p(x)} + |u|^{p(x)}] dx}{\int_{\partial\Omega} |u|^{p(x)} d\sigma_x} = \frac{\int_{B_\Omega(\partial\Omega, \delta)} [|\nabla tu_1|^p + |tu_1|^p] dx}{p \int_{\partial\Omega} \frac{1}{p} |tu_1|^p d\sigma_x} \geq \frac{\beta_{(\delta, 1)}}{p} > 0. \quad \square$$

By the absolute continuity of the integral (see [26]), we have

Lemma 4.3. Let $u \in X$ be given, for any $\delta > 0$, define $\beta_u(\delta) = \int_{B_\Omega(\partial\Omega, \delta)} \frac{1}{p(x)} [|\nabla u|^{p(x)} + |u|^{p(x)}] dx$. Then, $\forall \varepsilon > 0$, there exists $\delta_0 > 0$ such that for all $\delta \in (0, \delta_0)$, $\beta_u(\delta) < \varepsilon$.

Theorem 4.2. Suppose that there exist $\delta > 0$ and $x_0 \in \partial\Omega$ such that

- (1) $p(x) \equiv p_0, \forall x \in B_{\partial\Omega}(x_0, \delta)$;
- (2) $p(x) > p_0$ (or $p(x) < p_0$), $\forall x \in B_\Omega(x_0, \delta)$.

Then, $\lim_{\alpha \rightarrow 0} \lambda_{(1, \alpha)} = 0$ (or $\lim_{\alpha \rightarrow +\infty} \lambda_{(1, \alpha)} = 0$) and $\lambda_* = 0$.

Proof. We only prove the case of $p(x) > p_0$, the other is similar.

Let $u \in C^\infty(\Omega)$ and

$$u(x) = \begin{cases} 1, & x \in B_\Omega(x_0, \frac{\delta}{4}), \\ 0, & x \in \Omega \setminus B_\Omega(x_0, \frac{\delta}{2}). \end{cases}$$

By Lemma 4.3, $\forall \varepsilon > 0$, there exists $\delta_0 \in (0, \frac{\delta}{4})$ such that $\forall \delta_1 \in (0, \delta_0)$,

$$\int_{B_\Omega(\partial\Omega, \delta_1)} \frac{1}{p(x)} [|\nabla u|^{p(x)} + |u|^{p(x)}] dx < \frac{\varepsilon}{2c},$$

where $c = \frac{p^+}{p^- \int_{B_{\partial\Omega}(x_0, \frac{\delta}{2})} \frac{|u|^{p(x)}}{p(x)} d\sigma_x}$.

Since $p \in C(\overline{\Omega})$ and (2), $\forall x \in B_\Omega(x_0, \frac{\delta}{2}) \setminus B(\partial\Omega, \delta_1)$,

$$p(x) - p_0 \geq p^- \left(\overline{B_\Omega\left(x_0, \frac{\delta}{2}\right)} \setminus B(\partial\Omega, \delta_1) \right) - p_0 =: \varepsilon_0 > 0.$$

Obviously, for any $\alpha > 0$, there exists $t > 0$ such that $tu \in M_\alpha$ and $t \rightarrow 0$ as $\alpha \rightarrow 0$. Then there exists $\alpha_0 > 0$ such that $\forall \alpha \in (0, \alpha_0)$,

$$0 < t < \min \left\{ 1, \left(\frac{\varepsilon}{2cF(u)} \right)^{\frac{1}{\varepsilon_0}} \right\}.$$

Let u_0 be the eigenfunction associated with $\lambda_{(1,\alpha)}$, we have

$$\begin{aligned} \lambda_{(1,\alpha)} &\leq \frac{p^+ F(u_0)}{p^- G(u_0)} \leq \frac{p^+ F(tu)}{p^- \alpha} \\ &= \frac{p^+ \int_{B_\Omega(x_0, \frac{\delta}{2}) \setminus B_\Omega(\partial\Omega, \delta_1)} \frac{t^{p(x)}}{p(x)} [|\nabla u|^{p(x)} + |u|^{p(x)}] dx}{p^- \int_{B_{\partial\Omega}(x_0, \frac{\delta}{2})} \frac{t^{p(x)}}{p(x)} |u|^{p(x)} d\sigma_x} \\ &\quad + \frac{p^+ \int_{B_\Omega(\partial\Omega, \delta_1) \cap B_\Omega(x_0, \frac{\delta}{2})} \frac{t^{p(x)}}{p(x)} [|\nabla u|^{p(x)} + |u|^{p(x)}] dx}{p^- \int_{B_{\partial\Omega}(x_0, \frac{\delta}{2})} \frac{t^{p(x)}}{p(x)} |u|^{p(x)} d\sigma_x} \\ &= c \int_{B_\Omega(x_0, \frac{\delta}{2}) \setminus B_\Omega(\partial\Omega, \delta_1)} \frac{t^{p(x)-p_0}}{p(x)} [|\nabla u|^{p(x)} + |u|^{p(x)}] dx \\ &\quad + c \int_{B_\Omega(\partial\Omega, \delta_1) \cap B_\Omega(x_0, \frac{\delta}{2})} \frac{t^{p(x)-p_0}}{p(x)} [|\nabla u|^{p(x)} + |u|^{p(x)}] dx \\ &\leq c t^{\varepsilon_0} \int_{B_\Omega(x_0, \frac{\delta}{2}) \setminus B_\Omega(\partial\Omega, \delta_1)} \frac{1}{p(x)} [|\nabla u|^{p(x)} + |u|^{p(x)}] dx \\ &\quad + c \int_{B_\Omega(\partial\Omega, \delta_1)} \frac{1}{p(x)} [|\nabla u|^{p(x)} + |u|^{p(x)}] dx < \varepsilon. \quad \square \end{aligned}$$

Theorem 4.3. Suppose that there exist an open subset $V \subset \partial\Omega$, $\delta > 0$ and $\xi \in \mathbb{R}^N \setminus \overline{V}$ such that $\forall x \in V$, $I_x =: \{x + \tau \xi_x \mid \tau \in (0, \delta)\} \subset \Omega$, where $\xi_x = \frac{\xi - x}{|\xi - x|}$ or $\frac{x - \xi}{|\xi - x|}$,

(A) $p^-(V) < p^-(\partial V)$;

(B) $p(x) < p(y)$, $\forall y \in I_x$.

Then, $\lim_{\alpha \rightarrow 0} \lambda_{(1,\alpha)} = 0$ and $\lambda_* = 0$.

Proof. For any $c, d > 0$, denote by

$$p^c =: \{x \in V \mid p(x) < p^-(V) + c\},$$

$$p^{c \times d} =: \{x + \tau \xi_x \mid x \in p^c, \tau \in (0, d)\}.$$

Since (A), let $\varepsilon_0 = \frac{p^-(\partial V) - p^-(V)}{16}$. Then, $\overline{p^{4\varepsilon_0}} \subset p^{8\varepsilon_0}$ and we can take a function $u(x) \in C^\infty(\Omega)$ such that

$$u(x) = \begin{cases} 1, & x \in p^{4\varepsilon_0 \times \frac{\delta}{4}}, \\ 0, & x \in \Omega \setminus p^{8\varepsilon_0 \times \frac{\delta}{2}}. \end{cases}$$

For any $\varepsilon > 0$, let $\delta_1 = \min\{\frac{\varepsilon p^-}{2p^+}, \frac{\delta}{8}\}$. Since $p \in C(\overline{\Omega})$ and (B), $\forall x \in p^{8\varepsilon_0 \times \frac{\delta}{2}} \setminus p^{2\varepsilon_0 \times \delta_1}$, we have

$$p(x) - p^-(V) \geq p^-(\overline{p^{8\varepsilon_0 \times \frac{\delta}{2}} \setminus p^{2\varepsilon_0 \times \delta_1}}) - p^-(V) =: 2\varepsilon_1 > 0.$$

Similar to the proof of Theorem 4.2, there exists $\alpha_0 > 0$ such that $\forall \alpha \in (0, \alpha_0)$ and $G(tu) = \alpha$ with $t > 0$, we have

$$0 < t < \min\left\{1, \left(\frac{\varepsilon p^- \int_{p^{\varepsilon_1}} \frac{|u|^{p(x)}}{p(x)} d\sigma_x}{2p^+ F(u)}\right)^{\frac{1}{\varepsilon_1}}\right\}.$$

Then,

$$\frac{\int_{p^{8\varepsilon_0 \times \frac{\delta}{2}} \setminus p^{2\varepsilon_0 \times \delta_1}} \frac{1}{p(x)} [|\nabla tu|^{p(x)} + |tu|^{p(x)}] dx}{\int_{\partial\Omega} \frac{1}{p(x)} |tu|^{p(x)} d\sigma_x} \leq \frac{\int_{p^{8\varepsilon_0 \times \frac{\delta}{2}} \setminus p^{2\varepsilon_0 \times \delta_1}} \frac{t^{p^-(V)+2\varepsilon_1}}{p(x)} [|\nabla u|^{p(x)} + |u|^{p(x)}] dx}{\int_{p^{\varepsilon_1}} \frac{t^{p^-(V)+\varepsilon_1}}{p(x)} |u|^{p(x)} d\sigma_x} < \frac{\varepsilon p^-}{2p^+}.$$

On the other hand,

$$\begin{aligned} \frac{\int_{p^{2\varepsilon_0 \times \delta_1}} \frac{1}{p(x)} [|\nabla tu|^{p(x)} + |tu|^{p(x)}] dx}{\int_{\partial\Omega} \frac{1}{p(x)} |tu|^{p(x)} d\sigma_x} &< \frac{\int_{p^{2\varepsilon_0}} (\int_0^{\delta_1} \frac{t^{p(x+\tau\xi_x)}}{p(x+\tau\xi_x)} d\tau) d\sigma_x}{\int_{p^{2\varepsilon_0}} \frac{t^{p(x)}}{p(x)} d\sigma_x} \\ &\leq \frac{\int_{p^{2\varepsilon_0}} (\int_0^{\delta_1} \frac{t^{p(x)}}{p(x)} d\tau) d\sigma_x}{\int_{p^{2\varepsilon_0}} \frac{t^{p(x)}}{p(x)} d\sigma_x} = \delta_1 \leq \frac{\varepsilon p^-}{2p^+}. \end{aligned}$$

Thus,

$$\begin{aligned} \lambda_{(1,\alpha)} &\leq \frac{p^+ \int_{p^{8\varepsilon_0 \times \frac{\delta}{2}} \setminus p^{2\varepsilon_0 \times \delta_1}} \frac{1}{p(x)} [|\nabla tu|^{p(x)} + |tu|^{p(x)}] dx}{p^- \int_{\partial\Omega} \frac{1}{p(x)} |tu|^{p(x)} d\sigma_x} \\ &\quad + \frac{p^+ \int_{p^{2\varepsilon_0 \times \delta_1}} \frac{1}{p(x)} [|\nabla tu|^{p(x)} + |tu|^{p(x)}] dx}{p^- \int_{\partial\Omega} \frac{1}{p(x)} |tu|^{p(x)} d\sigma_x} < \varepsilon. \quad \square \end{aligned}$$

Since $f = \frac{a^t}{t}$ with $a > e$ is increasing with respect to $t \in (1, +\infty)$, similar to the proof of Theorem 4.3, we can prove that

Corollary 4.1. In Theorem 4.3, the conditions (A), (B) are replaced by

- (A⁺) $p^+(V) > p^+(\partial V)$;
 (B⁺) $p(x) > p(y), \forall y \in I_x$,

respectively. Then, $\lim_{\alpha \rightarrow +\infty} \lambda_{(1,\alpha)} = 0$ and $\lambda_* = 0$.

In Theorem 4.3, let $|\xi| \rightarrow +\infty$, then $\xi_x \rightarrow \eta$. Hence in the proof of Theorem 4.3, if each ξ_x is replaced by η , we can get

Corollary 4.2. *Let V and δ be the same as in Theorem 4.3. Suppose that there exists a vector $\eta \in \mathbb{R}^N \setminus \{0\}$ such that $\forall x \in V$, $I_x =: \{x + \tau\eta \mid \tau \in (0, \delta)\} \subset \Omega$. And the conditions (A) and (B) of Theorem 4.3 still hold. Then, $\lim_{\alpha \rightarrow 0} \lambda_{(1,\alpha)} = 0$ and $\lambda_* = 0$.*

Similar to Corollary 4.1, we can obtain the corollary of Corollary 4.2.

Corollary 4.3. *In Corollary 4.2, the conditions (A), (B) are replaced by (A^+) , (B^+) of Corollary 4.1, respectively. Then, $\lim_{\alpha \rightarrow +\infty} \lambda_{(1,\alpha)} = 0$ and $\lambda_* = 0$.*

Remark 4.2.

- (1) For the constant exponent case (i.e. $p(x) \equiv p = \text{constant}$), $\lambda_* = \lambda_{(1,\alpha)} = \lambda_1$ and λ_* is a principal eigenvalue.
- (2) For a variable exponent $p(x)$, suppose that under some assumptions, $\lambda_* = 0$. This means that under some conditions there does not exist a principal eigenvalue and the set of eigenvalues is not closed.
- (3) For a variable exponent $p(x)$ ($\neq \text{constant}$), suppose that under some assumptions, $\lambda_* > 0$. We do not know whether λ_* is an eigenvalue. Moreover, if λ_* is not an eigenvalue, then, there does not exist a principal eigenvalue and the set of eigenvalues is not closed; if λ_* is an eigenvalue, then, $\lambda_1 = \lambda_*$ is a principal eigenvalue but this first eigenvalue λ_1 may not be isolated.

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